2 - 10 ODEs reducible to Bessel's ODE
This is just a sample of such ODEs; some more follow in the next problem set. Find a general solution in terms of J_ν and $J_{-\nu}$ or indicate when this is not possible. Use the indicated substitutions.

3. $x y'' + y' + \frac{1}{4} y = 0 \quad (\sqrt{x} = z)$

```
ClearAll["Global`*"]
```

```
e1 = {x y''[x] + y'[x] + 1/4 y[x] == 0}
```

$\left\{\frac{y[x]}{4} + y'[x] + x y''[x] == 0\right\}$

```
e2 = DSolve[e1, y, x, Assumptions → {√x → z}]
```

```
{{y → Function[{x}, BesselJ[0, √x] C[1] + 2 BesselY[0, √x] C[2]]}}
```

Since Bessel is special function, it takes **FullSimplify** to simplify it, thereby checking the Mathematica answer.

```
e1 /. e2 // FullSimplify
{{True}}
```

The yellow expression above seems to include the text answer, but also a BesselY, which the text answer does not mention. I do not know when, if ever, the BesselY would be useful, but it only takes assigning C[2] to zero to make it go away, leaving the text answer.

```
e2[[1, 1, 2]]
```

```
Function[{x}, BesselJ[0, √x] C[1] + 2 BesselY[0, √x] C[2]]
```

```
gad = e2[[1, 1, 2]] /. {C[1] → 1, C[2] → 1}
```

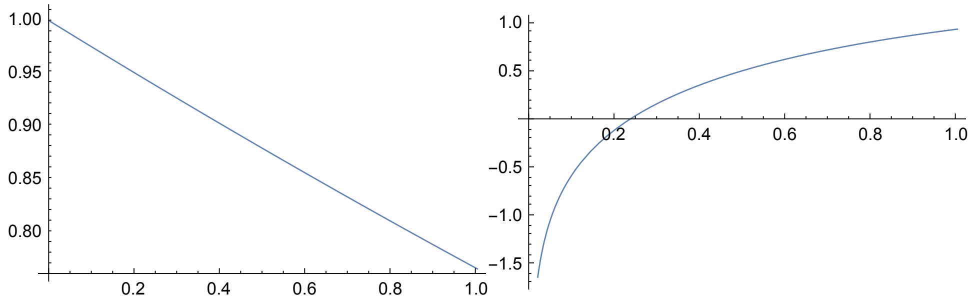
```
Function[{x}, BesselJ[0, √x] 1 + 2 BesselY[0, √x]]
```

```
pl1 = Plot[BesselJ[0, √x], {x, 0, 1}, PlotRange → Automatic,
   PlotStyle → Thickness[0.003], ImageSize → 250];
```

I have to admit, the below plots show that the BesselY does seem to add something to the situation.

```
pl2 = Plot[BesselJ[0, √x] 1 + 2 BesselY[0, √x], {x, 0, 1},
   PlotRange → Automatic, PlotStyle → Thickness[0.003], ImageSize → 250];
```

```
Row[{pl1, pl2}]
```



5. Two – parameter ODE
$x^2 y'' + x y' + \left(\lambda^2 x^2 - \nu^2\right) y = 0 \quad (\lambda x = z)$

```
ClearAll["Global`*"]
```

```
e1 = {x² y''[x] + x y'[x] + (λ² x² - ν²) y[x] == 0}
```

```
{(x² λ² - ν²) y[x] + x y′[x] + x² y″[x] == 0}
```

```
e2 = DSolve[e1, y, x, Assumptions → {λ x → z}]
```

```
{{y → Function[{x}, BesselJ[ν, x λ] C[1] + BesselY[ν, x λ] C[2]]}}
```

```
e1 /. e2 // FullSimplify
```

```
{{True}}
```

The yellow cell above contains a Bessel function of the first kind and one of the second kind, which *Mathematica* has checked and found correct. The text answer contains the same Bessel of the first kind, but the other half is expressed as one with negative first argument, as shown in the unsuccessful PZQ below.

```
PossibleZeroQ[BesselJ[ν, x λ] C[1] + BesselY[ν, x λ] C[2] -
  (BesselJ[ν, x λ] C[1] + (-1)^ν BesselJ[ν, x λ] C[2])]
```

```
False
```

I try to verify numbered line (25) on p. 194 and find that it doesn't work. Wonder if there is a possibility of misinterpretation.

```
PossibleZeroQ[BesselJ[-ν, x λ] - (-1)^ν BesselJ[ν, x λ] ]
```

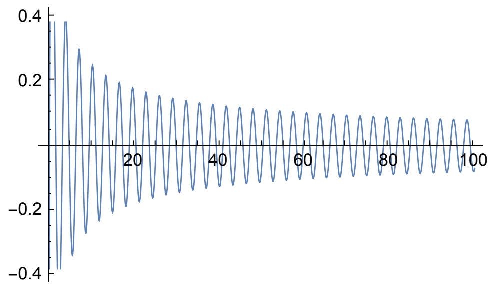
```
False
```

The Mathematica answer is plotted.

```
glif = e2[[1, 1]] /. {C[1] → 1, C[2] → 1, λ → 2, ν → 0}
```

```
y → Function[{x}, BesselJ[0, x 2] 1 + BesselY[0, x 2] 1]
```

```
Plot[y[x] /. glif, {x, -100, 100}, PlotRange → Automatic,
 PlotStyle → Thickness[0.003], ImageSize → 250]
```



7. $x^2 y'' + x y' + \frac{1}{4}\left(x^2 - 1\right) y = 0 \quad (x = 2 z)$

```
ClearAll["Global`*"]
```

e1 = $\left\{x^2 y''[x] + x y'[x] + \frac{1}{4}\left(x^2 - 1\right) y[x] == 0\right\}$

$\left\{\frac{1}{4}\left(-1 + x^2\right) y[x] + x y'[x] + x^2 y''[x] == 0\right\}$

```
e2 = DSolve[e1, y[x], x, Assumptions → {x → 2 z, x ∈ Reals}]
```

$\left\{\left\{y[x] \to \frac{e^{-\frac{i x}{2}} C[1]}{\sqrt{x}} - \frac{i e^{\frac{i x}{2}} C[2]}{\sqrt{x}}\right\}\right\}$

```
e3 = ExpToTrig[e2]
```

$\left\{\left\{y[x] \to \frac{C[1] \operatorname{Cos}\left[\frac{x}{2}\right]}{\sqrt{x}} - \frac{i C[2] \operatorname{Cos}\left[\frac{x}{2}\right]}{\sqrt{x}} - \frac{i C[1] \operatorname{Sin}\left[\frac{x}{2}\right]}{\sqrt{x}} + \frac{C[2] \operatorname{Sin}\left[\frac{x}{2}\right]}{\sqrt{x}}\right\}\right\}$

Mathematica gives the right answer, but includes imaginary elements, which, looking at the text answer, are not necessary.

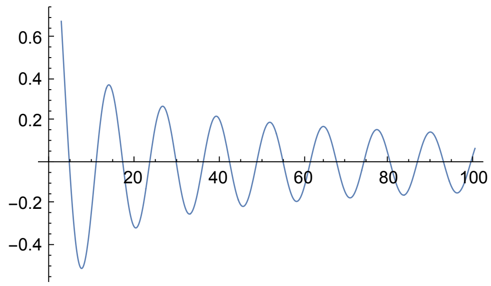
e4 = e3 /. $\left\{-\frac{i C[2] \operatorname{Cos}\left[\frac{x}{2}\right]}{\sqrt{x}} \to 0,\ -\frac{i C[1] \operatorname{Sin}\left[\frac{x}{2}\right]}{\sqrt{x}} \to 0\right\}$

$\left\{\left\{y[x] \to \frac{C[1] \operatorname{Cos}\left[\frac{x}{2}\right]}{\sqrt{x}} + \frac{C[2] \operatorname{Sin}\left[\frac{x}{2}\right]}{\sqrt{x}}\right\}\right\}$

```
e5 = e4 /. {C[1] → 1, C[2] → 1}
```

$\left\{\left\{y[x] \to \frac{\operatorname{Cos}\left[\frac{x}{2}\right]}{\sqrt{x}} + \frac{\operatorname{Sin}\left[\frac{x}{2}\right]}{\sqrt{x}}\right\}\right\}$

```
Plot[y[x] /. e5, {x, -100, 100}, PlotRange → Automatic,
 PlotStyle → Thickness[0.003], ImageSize → 250]
```



9. x y'' + (2 ν + 1) y' + x y = 0 (y = x^{-ν} u)

```
ClearAll["Global`*"]
e1 = {x y''[x] + (2 ν + 1) y'[x] + x y[x] == 0}
{x y[x] + (1 + 2 ν) y′[x] + x y″[x] == 0}
e2 = DSolve[e1, y, x, Assumptions → {y[x] → x^-ν u}]
{{y → Function[{x}, x^-ν BesselJ[ν, x] C[1] + x^-ν BesselY[ν, x] C[2]]}}
```

Mathematica's answer checks.

```
e1 /. e2 // FullSimplify
{{True}}
```

The form of the above answer is not exactly the same as the text answer. They do not seem to be the same.

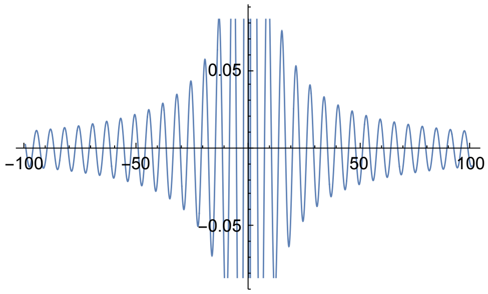
```
PossibleZeroQ[x^-ν BesselJ[ν, x] C[1] + x^-ν BesselY[ν, x] C[2] -
  (x^-ν BesselJ[ν, x] C[1] + x^-ν BesselJ[-ν, x] C[2])]
False
```

Manipulating the Mathematica answer for a plot.

```
e3 = e2 /. {C[1] → 1, C[2] → 1}
{{y[x] → x^-ν BesselJ[ν, x] + x^-ν BesselY[ν, x]}}
e4 = Simplify[e3, Assumptions → ν ∉ Integers]
{{y[x] → x^-ν (BesselJ[ν, x] + BesselY[ν, x])}}
e5 = e4 /. ν → .5
```

$$\left\{\left\{y[x] \to \frac{\frac{0.7978845608028654\, \mathrm{Cos}[1.5708-x]}{\sqrt{x}} - \frac{0.7978845608028654\, \mathrm{Sin}[1.5708-x]}{\sqrt{x}}}{x^{0.5}}\right\}\right\}$$

```
Plot[y[x] /. e5, {x, -100, 100}, PlotRange → Automatic,
 PlotStyle → Thickness[0.003], ImageSize → 250]
```



13 -15 Zeros of Bessel functions play a key role in modeling (e.g. of vibrations; see section 12.9).

13. Interlacing of zeros. Using numbered line (21), p. 191, and Rolle's theorem, show that between any two consecutive positive zeros of $J_n(x)$ there is precisely one zero of $J_{n+1}(x)$.

15. Interlacing of zeros. Using numbered line (21) and Rolle's theorem, show that between any two consecutive zeros of $J_0(x)$ there is precisely one zero of $J_1(x)$.

19 - 25 Application of (21): deriviatives, integrals
Use the powerful formulas (21) to do problems 19 - 25.

23. Integration. Show that Integrate$[x^2 J_0, x] = x^2 J_1[x] + x J_0[x] -$ Integrate$[J_0[x], x]$. (The last integral is nonelementary; tables exist, e.g. in reference [A13] in appendix 1.)

```
a1 = Integrate[x^2 BesselJ[0, x], x]
```

$$\frac{1}{3} x^3 \text{HypergeometricPFQ}\left[\left\{\frac{3}{2}\right\}, \left\{1, \frac{5}{2}\right\}, -\frac{x^2}{4}\right]$$

```
a2 = x^2 BesselJ[1, x]
```

x^2 BesselJ[1, x]

```
a3 = x BesselJ[0, x]
```

x BesselJ[0, x]

```
a4 = Integrate[BesselJ[0, x], x]
```

$$x \text{HypergeometricPFQ}\left[\left\{\frac{1}{2}\right\}, \left\{1, \frac{3}{2}\right\}, -\frac{x^2}{4}\right]$$

```
PossibleZeroQ[a1 - (a2 + a3 - a4)]
False
```

The demonstration fails. However, I don't know if I expressed everything right.